

A Generalization of the Fibonacci Word Fractal and the Fibonacci Snowflake

José L. Ramírez ^{*1}, Gustavo N. Rubiano ^{†2}, and Rodrigo de Castro^{‡2}

¹Escuela de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia

²Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

January 16, 2013

Abstract

In this paper we introduce a family of infinite words that generalize the Fibonacci word and we study their combinatorial properties. Moreover, we associate to this family of words a family of curves, which have fractal properties, in particular these curves have as attractor the Fibonacci word fractal. Finally, we describe an infinite family of polyominoes (double squares) from the generalized Fibonacci words and we study some of their geometric properties. These last polyominoes generalize the Fibonacci snowflake.

Keywords: Combinatorics on words, Fibonacci word, Fibonacci word fractal, Fibonacci snowflake, Polyomino, Tessellation.

MSC: 05B50, 11B39, 28A80, 68R15.

1 Introduction

The infinite Fibonacci word,

$$\mathbf{f} = 0100101001001010010100100101\dots$$

is certainly one of the most studied words in the combinatorics on words, e.g. [10], [11], [12], [13], [15]. It is the archetype of a Sturmian word [14]. This word can be associated with a curve, which has fractal properties, which are obtained from combinatorial properties of \mathbf{f} , [3], [16]. This word can also be associated to a family of polyominoes, which tiles the plane by translation. They are double squares and are called Fibonacci snowflakes, [3], [6].

In this paper we introduce a family of words $\mathbf{f}^{[i]}$ (Definition 8) that generalize the Fibonacci word, which are characteristic words of slope $\frac{i-\phi}{i^2-i-1}$ where ϕ is the golden ratio (Theorem 1). From this family we define a family of curves, which have as attractor the Fibonacci word fractal (Proposition 7) and have the same properties (Proposition 6). Finally, we study a family of polyominoes which generalize the Fibonacci snowflake and we study their geometric properties, such as perimeter (Proposition 13) and area (Proposition 14), which is related to generalized Pell numbers. This polyominoes are also double squares (Theorem 2) and have the same fractal dimension of the Fibonacci word Fractal.

^{*}josel.ramirez@ima.usergioarboleda.edu.co

[†]gnrubiano@unal.edu.co

[‡]rdcastro@unal.edu.co

2 Definitions and Notation

The terminology and notations are mainly those of Lothaire [14] and Allouche and Shallit [1]. Let Σ be a finite alphabet, whose elements are called symbols. A word over Σ is a finite sequence of symbols from Σ . The set of all words over Σ , i.e., the free monoid generated by Σ , is denoted by Σ^* . The identity element ϵ of Σ^* is called the empty word. For any word $w \in \Sigma^*$, $|w|$ denotes its length, i.e., the number of symbols occurring in w . The length of ϵ is taken to be equal to 0. If $a \in \Sigma$ and $w \in \Sigma^*$, then $|w|_a$ denotes the number of occurrences of a in w .

For two words $u = a_1a_2 \cdots a_k$ and $v = b_1b_2 \cdots b_s$ in Σ^* , we denote by uv the concatenation of the two words, that is, $uv = a_1a_2 \cdots a_kb_1b_2 \cdots b_s$. If $v = \epsilon$ then $u\epsilon = \epsilon u = u$, moreover, by u^n we denote the word $uu \cdots u$ (n times). A word v is a subword, or factor, of u if there exist $x, y \in \Sigma^*$ such that $u = xvy$. If $x = \epsilon$ ($y = \epsilon$), then v is called prefix (suffix) of u .

The reversal of a word $u = a_1a_2 \cdots a_n$, is the word $u^R = a_n \cdots a_2a_1$ and $\epsilon^R = \epsilon$. A word u is a palindrome if $u^R = u$.

An infinite word over Σ is a map $\mathbf{u} : \mathbb{N} \rightarrow \Sigma$. It is written $\mathbf{u} = a_1a_2a_3 \dots$. The set of all infinite words over Σ is denoted by Σ^ω .

Example 1. *The following is an example of an infinite word $\mathbf{p} = (p_n)_{n \geq 1} = 0110101000101 \dots$, where $p_n = 1$ if n is a prime number and $p_n = 0$ otherwise. The word \mathbf{p} is called the characteristic sequence of the prime numbers.*

Definition 1. *Let Σ and Δ be alphabets. A morphism is a map $h : \Sigma^* \rightarrow \Delta^*$ such that $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma^*$. Is clear that $h(\epsilon) = \epsilon$.*

There is a special class of words, with many remarkable properties, the so-called Sturmian words. These words admit several equivalent definitions, (see, e.g. [1] or [14]).

Definition 2. *Let $\mathbf{w} \in \Sigma^\omega$. We define $P(\mathbf{w}, n)$, the complexity function of \mathbf{w} , to be the map that counts, for all integer $n \geq 0$, the number of subwords of length n in \mathbf{w} . An infinite word \mathbf{w} is a Sturmian word if $P(\mathbf{w}, n) = n + 1$ for all integer $n \geq 0$.*

Since $P(\mathbf{w}, 1) = 2$, any Sturmian word is over two symbols. The word \mathbf{p} , in the example 1, is not a Sturmian word because $P(\mathbf{p}, 2) = 4$.

Given two real numbers $\alpha, \beta \in \mathbb{R}$ with α irrational and $0 < \alpha < 1$, $0 \leq \beta < 1$, we define the infinite word $\mathbf{w} = w_1w_2w_3 \cdots$ as

$$w_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor.$$

The numbers α and β are the slope and the intercept, respectively. This word is called mechanical. The mechanical words are equivalent to Sturmian words, [14]. As special case, when $\beta = 0$, we obtain the called characteristic words.

Definition 3. *Let α be an irrational number with $0 < \alpha < 1$. For $n \geq 1$, define*

$$w_\alpha(n) := \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$$

and

$$\mathbf{w}(\alpha) := w_\alpha(1)w_\alpha(2)w_\alpha(3) \cdots$$

then $\mathbf{w}(\alpha)$ is called a characteristic word with slope α .

On the other hand, note that every irrational $\alpha \in (0, 1)$ has a unique continued fraction expansion

$$\alpha = [0, a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where each a_i is a positive integer. Let $\alpha = [0, 1 + d_1, d_2, \dots]$ be an irrational number with $d_1 \geq 0$ and $d_n > 0$ for $n > 1$. To the directive sequence $(d_1, d_2, \dots, d_n, \dots)$, we associate a sequence $(s_n)_{n \geq -1}$ of words defined by

$$s_{-1} = 1, \quad s_0 = 0, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad (n \geq 1)$$

Such a sequence of words is called a standar sequence. This sequence is related to characteristic words in the followin way. Observe that, for any $n \geq 0$, s_n is a prefix of s_{n+1} , which gives meaning to $\lim_{n \rightarrow \infty} s_n$ as in infinite word. In fact, one can prove [14] that each s_n is a prefix of $w(\alpha)$ for all $n \geq 0$ and

$$w(\alpha) = \lim_{n \rightarrow \infty} s_n. \quad (1)$$

2.1 Fibonacci Word and Its Fractal Curve

Definition 4. *Fibonacci words are words over $\{0, 1\}$ defined inductively as follows*

$$f_0 = 1, \quad f_1 = 0, \quad f_n = f_{n-1} f_{n-2},$$

for $n \geq 2$. The words f_n are referred to as the finite Fibonacci words. The limit

$$\mathbf{f} = \lim_{n \rightarrow \infty} f_n = 0100101001001010010100100101 \dots$$

is called the Fibonacci word.

It is clear that $|f_n| = F_n$, where F_n is the n -th Fibonacci number¹. The infinite Fibonacci word \mathbf{f} is a Sturmian word [14], exactly, $\mathbf{f} = w\left(\frac{1}{\phi^2}\right)$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Definition 5. *The Fibonacci morphism $\sigma : \{0, 1\} \rightarrow \{0, 1\}$ is defined by $\sigma(0) = 01$ and $\sigma(1) = 0$.*

The Fibonacci word \mathbf{f} satisfies that $\lim_{n \rightarrow \infty} \sigma^n(1) = \mathbf{f}$.

Definition 6. *Let $\Phi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be a map such that Φ deletes the last two symbols, i.e., $\Phi(a_1 a_2 \dots a_n) = a_1 a_2 \dots a_{n-2}$, ($n \geq 2$).*

The following proposition summarizes some basic properties about Fibonacci word.

Proposition 1. *The Fibonacci word and the finite Fibonacci words, satisfy that*

- a. *The words 11 and 000 are not subwords of the Fibonacci word.*
- b. *Let ab be the last two symbols of f_n . For $n \geq 2$, we have $ab = 01$ if n is even and $ab = 10$ if n is odd.*
- c. *The concatenation of two successive Fibonacci words is “almost commutative”, i.e., $f_n f_{n-1}$ and $f_{n-1} f_n$ have a common prefix the length $F_n - 2$ for all $n \geq 2$.*
- d. *$\Phi(f_n)$ is a palindrome for all $n \geq 2$.*
- e. *For all $n \geq 6$, $f_n = f_{n-3} f_{n-3} f_{n-6} l_{n-3} l_{n-3}$, where $l_n = \Phi(f_n) b a$, i.e., l_n exchanges the two last symbols of f_n .*

Proposition 1 allows to establish some properties about the structure of the fractal curve, (see Proposition 2).

¹The Fibonacci number F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all integer $n \geq 2$ and with initial values $F_0 = 1 = F_1$.

2.2 The Fibonacci Word Fractal

The Fibonacci word can be associated to a curve from a drawing rule. We must travel the word in a particular way, depending on the symbol read a particular action is produced, this idea is the same as that used in the L-Systems [17]. In this case, the drawing rule is called “odd-even drawing rule” [16], this is defined as shown in the following table.

Symbol	Action
1	Draw a line forward.
0	Draw a line forward and if the symbol 0 is in a position even then turn left and if 0 is in a position odd then turn right.

Definition 7. The n th-curve of Fibonacci, denoted by \mathcal{F}_n , is obtained by applying the odd-even drawing rule to the word f_n . The Fibonacci word fractal \mathcal{F} , is defined as

$$\mathcal{F} = \lim_{n \rightarrow \infty} \mathcal{F}_n.$$

Example 2. In the figure 1 we show the curve \mathcal{F}_{10} and \mathcal{F}_{17} . The graphs in this paper were generated using the software **Mathematica** 8.0, [18].

[illegible]

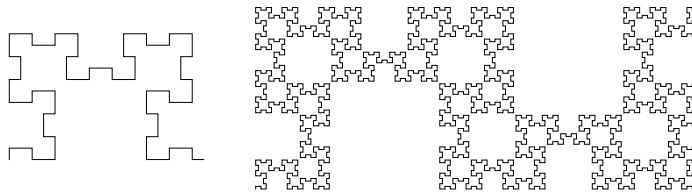


Figure 1: Fibonacci curves \mathcal{F}_{10} and \mathcal{F}_{17} corresponding to the words f_{10} and f_{17} .

In the next proposition we show some properties of the curves \mathcal{F}_n and \mathcal{F} . It comes directly from the properties of the Fibonacci word, see Proposition 1.

Proposition 2. *Fibonacci word fractal \mathcal{F} and the curve \mathcal{F}_n have the following properties:*

- \mathcal{F} is composed only of segments of length 1 or 2.
- The \mathcal{F}_n curve is similar to the curve \mathcal{F}_{n-3} .
- The curve \mathcal{F}_n is symmetric.
- The \mathcal{F}_n curve is composed by 5 curves: $\mathcal{F}_n = \mathcal{F}_{n-3}\mathcal{F}_{n-3}\mathcal{F}_{n-6}\mathcal{F}'_{n-3}\mathcal{F}'_{n-3}$, where \mathcal{F}'_n is obtained by applying the odd-even drawing rule to word l_n .
- The fractal dimension of the Fibonacci word fractal is

$$3 \frac{\log \phi}{\log(1 + \sqrt{2})} = 1.6379 \dots$$

More of these properties can be found in [16].

3 Generalized Fibonacci Words and Fibonacci Word Fractals

In this section, we introduce a generalization of the Fibonacci word and the Fibonacci word fractal, and we show that the propositions 1 and 2 remain.

Definition 8. The (n, i) -Fibonacci words are words over $\{0, 1\}$ defined inductively as follows

$$f_0^{[i]} = 0, \quad f_1^{[i]} = 0^{i-1}1, \quad f_n^{[i]} = f_{n-1}^{[i]}f_{n-2}^{[i]},$$

for all $n \geq 2$ and $i \geq 1$. The infinite word

$$\mathbf{f}^{[i]} := \lim_{n \rightarrow \infty} f_n^{[i]}$$

is called the i -Fibonacci word.

For $i = 2$ we have the Fibonacci word.

Example 3. The first i -Fibonacci words are

$$\begin{aligned} \mathbf{f}^{[1]} &= 1011010110110\cdots = \overline{\mathbf{f}}, & \mathbf{f}^{[2]} &= 0100101001001\cdots = \mathbf{f}, & \mathbf{f}^{[3]} &= 0010001001000\cdots, \\ \mathbf{f}^{[4]} &= 0001000010001\cdots, & \mathbf{f}^{[5]} &= 0000100000100\cdots, & \mathbf{f}^{[6]} &= 0000010000001\cdots \end{aligned}$$

The following proposition relates the Fibonacci word \mathbf{f} with $\mathbf{f}^{[i]}$.

Proposition 3. Let $\varphi_i : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be the morphism defined by $\varphi_i(0) = 0$ and $\varphi_i(1) = 0^i 1$, $i \geq 0$, then

$$\mathbf{f}^{[i+2]} = \varphi_i(\mathbf{f})$$

for all $i \geq 0$.

Proof. It suffices to prove that $f_{n-1}^{[i+2]} = \varphi_i(f_n)$ for all integer $n \geq 2$ and $i \geq 0$. We prove this by induction on n . For $n = 2$ we have $\varphi_i(f_2) = \varphi_i(01) = 0^{i+1}1 = f_1^{[i+2]}$. Now suppose the result is true for n . Then $\varphi_i(f_{n+1}) = \varphi_i(f_n f_{n-1}) = \varphi_i(f_n) \varphi_i(f_{n-1}) = f_{n-1}^{[i+2]} f_{n-2}^{[i+2]} = f_n^{[i+2]}$. \square

Definition 9. The (n, i) -th Fibonacci number $F_n^{[i]}$ is defined recursively by $F_0^{[i]} = 1$, $F_1^{[i]} = i$ and $F_n^{[i]} = F_{n-1}^{[i]} + F_{n-2}^{[i]}$ for all $n \geq 2$ and $i \geq 1$.

For $i = 1, 2$ we have the Fibonacci numbers. The table 1 shows the first numbers $F_n^{[i]}$ and their coincidence with some remarkable sequences in the OIES ².

i	$\left\{ F_n^{[i]} \right\}_{n \geq 0}$
1	$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}$, (A000045).
2	$\{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots\}$, (A000045).
3	$\{1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \dots\}$, (A000204).
4	$\{1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, \dots\}$, (A000285).
5	$\{1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309, 500, \dots\}$, (A022095).
6	$\{1, 6, 7, 13, 20, 33, 53, 86, 139, 225, 364, 589, \dots\}$, (A022096).

Table 1: First numbers $F_n^{[i]}$.

Nota that the length of the word $f_n^{[i]}$ is the (n, i) -th Fibonacci number $F_n^{[i]}$, i.e., $|f_n^{[i]}| = F_n^{[i]}$. It is clear because $f_n^{[i]} = f_{n-1}^{[i]} f_{n-2}^{[i]}$ then $|f_n^{[i]}| = |f_{n-1}^{[i]}| + |f_{n-2}^{[i]}|$, moreover $|f_0^{[i]}| = 1$ and $|f_1^{[i]}| = i$.

² Many integer sequences and their properties are to be found electronically on the On-Line Encyclopedia of Sequences, [19].

Proposition 4. A formula for the (n, i) -th Fibonacci number is

$$F_n^{[i]} = \frac{1}{2\sqrt{5}} \left(\left(\frac{1-\sqrt{5}}{2} \right)^n (\sqrt{5} + 1 - 2i) + \left(\frac{1+\sqrt{5}}{2} \right)^n (\sqrt{5} - 1 + 2i) \right).$$

Proof. The proof is by induction on n . This is clearly true for $n = 0, 1$. Now suppose the result is true for n . Then

$$F_{n+1}^{[i]} = F_n^{[i]} + F_{n-1}^{[i]} = \frac{1}{2\sqrt{5}} \left((\phi_1^n + \phi_1^{n-1}) (\sqrt{5} + 1 - 2i) + (\phi_2^n + \phi_2^{n-1}) (\sqrt{5} - 1 + 2i) \right)$$

where $\phi_1 = \frac{1-\sqrt{5}}{2}$ and $\phi_2 = \frac{1+\sqrt{5}}{2}$. Moreover

$$\phi_1^n + \phi_1^{n-1} = \phi_1^{n-1}(\phi_1 + 1) = \phi_1^{n-1} \left(\frac{1-\sqrt{5}}{2} + 1 \right) = \phi_1^{n-1} \phi_1^2 = \phi_1^{n+1}$$

analogously $\phi_2^n + \phi_2^{n-1} = \phi_2^{n+1}$. So

$$F_{n+1}^{[i]} = \frac{1}{2\sqrt{5}} \left(\phi_1^{n+1} (\sqrt{5} + 1 - 2i) + \phi_2^{n+1} (\sqrt{5} - 1 + 2i) \right). \quad \square$$

The following proposition generalizes the Proposition 1.

Proposition 5. The i -Fibonacci word and the (n, i) -Fibonacci words, satisfy that

- a. The word **11** is not a subword of the i -Fibonacci word, $i \geq 2$.
- b. Let ab be the last two symbols of $f_n^{[i]}$. For $n \geq 1$, we have $ab = 10$ if n is even and $ab = 01$ if n is odd, $i \geq 2$.
- c. The concatenation of two successive i -Fibonacci words is “almost commutative”, i.e., $f_{n-1}^{[i]} f_{n-2}^{[i]}$ and $f_{n-2}^{[i]} f_{n-1}^{[i]}$ have a common prefix the length $F_n^{[i]} - 2$ for all $n \geq 2$ and $i \geq 2$.
- d. $\Phi(f_n^{[i]})$ is a palindrome for all $n \geq 1$.
- e. For all $n \geq 6$, $f_n^{[i]} = f_{n-3}^{[i]} f_{n-3}^{[i]} f_{n-6}^{[i]} l_{n-3}^{[i]} l_{n-3}^{[i]}$, where $l_n^{[i]} = \Phi(f_n^{[i]})ba$.

Proof. a. It suffices to prove that **11** is not a subword of $f_n^{[i]}$, for $n \geq 0$. By induction on n . For $n = 0, 1$ it is clear. Assume for all $j < n$; we prove it for n . We know that $f_n^{[i]} = f_{n-1}^{[i]} f_{n-2}^{[i]}$ so by the induction hypothesis we have that **11** is not a subword of $f_{n-1}^{[i]}$ and $f_{n-2}^{[i]}$. Therefore, the only possibility is that **1** is a suffix of $f_{n-1}^{[i]}$ and **1** is a prefix of $f_{n-2}^{[i]}$, but this is impossible.

b. It is clear by induction on n .

c. By definition of $f_n^{[i]}$, we have

$$\begin{aligned} f_{n-1}^{[i]} f_{n-2}^{[i]} &= f_{n-2}^{[i]} f_{n-3}^{[i]} \cdot f_{n-3}^{[i]} f_{n-4}^{[i]} = f_{n-3}^{[i]} f_{n-4}^{[i]} \cdot f_{n-3}^{[i]} f_{n-3}^{[i]} f_{n-4}^{[i]}, \\ f_{n-2}^{[i]} f_{n-1}^{[i]} &= f_{n-3}^{[i]} f_{n-4}^{[i]} \cdot f_{n-2}^{[i]} f_{n-3}^{[i]} = f_{n-3}^{[i]} f_{n-4}^{[i]} \cdot f_{n-3}^{[i]} f_{n-4}^{[i]} \cdot f_{n-3}^{[i]}. \end{aligned}$$

Hence the words have a common prefix of length $F_{n-3}^{[i]} + F_{n-4}^{[i]} + F_{n-3}^{[i]}$. By the induction hypothesis $f_{n-3}^{[i]} f_{n-4}^{[i]}$ and $f_{n-4}^{[i]} f_{n-3}^{[i]}$ have common prefix of length $F_{n-2}^{[i]} - 2$. Therefore the words have a common prefix of length

$$2F_{n-3}^{[i]} + F_{n-4}^{[i]} + F_{n-2}^{[i]} - 2 = F_{n-2}^{[i]} + F_{n-1}^{[i]} - 2 = F_n^{[i]} - 2.$$

d. By induction on n . If $n = 2$ then $\Phi(f_2^{[i]}) = 0^i$. Now suppose that the result is true for all $j < n$; we prove it for n . Then

$$(\Phi(f_n^{[i]}))^R = (\Phi(f_{n-1}^{[i]}f_{n-2}^{[i]}))^R = (f_{n-1}^{[i]}\Phi(f_{n-2}^{[i]}))^R = \Phi(f_{n-2}^{[i]})^R(f_{n-1}^{[i]})^R = \Phi(f_{n-2}^{[i]})(f_{n-1}^{[i]})^R,$$

If n is even then $f_n^{[i]} = \Phi(f_n^{[i]})10$ and

$$\Phi(f_n^{[i]}) = \Phi(f_{n-2}^{[i]})(\Phi(f_{n-1}^{[i]})01)^R = \Phi(f_{n-2}^{[i]})10\Phi(f_{n-1}^{[i]})^R = f_{n-2}^{[i]}\Phi(f_{n-1}^{[i]}) = \Phi(f_n^{[i]}).$$

If n is odd, the proof is analogous.

e. By definition of $f_n^{[i]}$, we have

$$\begin{aligned} f_n^{[i]} &= f_{n-1}^{[i]}f_{n-2}^{[i]} \\ &= (f_{n-2}^{[i]}f_{n-3}^{[i]})(f_{n-3}^{[i]}f_{n-4}^{[i]}) \\ &= (f_{n-3}^{[i]}f_{n-4}^{[i]})(f_{n-4}^{[i]}f_{n-5}^{[i]})f_{n-3}^{[i]}f_{n-4}^{[i]} \\ &= f_{n-3}^{[i]}f_{n-4}^{[i]}(f_{n-5}^{[i]}f_{n-6}^{[i]})f_{n-5}^{[i]}(f_{n-4}^{[i]}f_{n-5}^{[i]})f_{n-4}^{[i]} \\ &= f_{n-3}^{[i]}(f_{n-4}^{[i]}f_{n-5}^{[i]})f_{n-6}^{[i]}(f_{n-5}^{[i]}f_{n-4}^{[i]})(f_{n-5}^{[i]}f_{n-4}^{[i]}) \\ &= f_{n-3}^{[i]}f_{n-3}^{[i]}f_{n-6}^{[i]}f_{n-3}^{[i]}f_{n-3}^{[i]}. \end{aligned}$$

□

Theorem 1. Let $\alpha = [0, i, \overline{1}]$ be an irrational number, with i a positive integer, then

$$w(\alpha) = f^{[i]}.$$

Proof. Let $\alpha = [0, i, \overline{1}]$ an irrational number, then its associated standar sequence is

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = s_0^{i-1}s_{-1} = 0^{i-1}1 \text{ and } s_n = s_{n-1}s_{n-2}, \quad n \geq 2.$$

Hence $\{s_n\}_{n \geq 0} = \{f_n^{[i]}\}_{n \geq 0}$ and by the equation (1), we have

$$w(\alpha) = \lim_{n \rightarrow \infty} s_n = f^{[i]}.$$

□

Remark. Note that

$$[0, i, \overline{1}] = \frac{1}{i + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}} = \frac{i - \phi}{i^2 - i - 1}$$

where ϕ is the golden ratio.

From the above theorem, we conclude that i -Fibonacci words are Sturmian words.

3.1 Generalized i -Fibonacci Word Fractal.

Definition 10. The (n, i) th-curve of Fibonacci, denoted by $\mathcal{F}_n^{[i]}$, is obtained by applying the odd-even drawing rule to the word $f_n^{[i]}$. The i -Fibonacci word fractal \mathcal{F} , is defined as

$$\mathcal{F}^{[i]} = \lim_{n \rightarrow \infty} \mathcal{F}_n^{[i]}.$$

In the table 2, we show the curves $\mathcal{F}_{16}^{[i]}$, for $i = 1, 2, 3, 4, 5$ and 6.

The following proposition generalizes the proposition 2 .

$\mathcal{F}_{16}^{[1]}$	$\mathcal{F}_{16}^{[2]}$	$\mathcal{F}_{16}^{[3]}$
$\mathcal{F}_{16}^{[4]}$	$\mathcal{F}_{16}^{[5]}$	$\mathcal{F}_{16}^{[6]}$

Table 2: Curves $\mathcal{F}_{16}^{[i]}$, for $i = 1, 2, 3, 4, 5$ and 6 .

Proposition 6. *The i -Fibonacci word fractal and the curve $\mathcal{F}_n^{[i]}$ have the following properties:*

- a. *The Fibonacci fractal $\mathcal{F}^{[i]}$ is composed only of segments of length 1 or 2.*
- b. *The $\mathcal{F}_n^{[i]}$ curve is similar to the curve $\mathcal{F}_{n-3}^{[i]}$.*
- c. *The $\mathcal{F}_n^{[i]}$ curve is composed by 5 curves: $\mathcal{F}_n^{[i]} = \mathcal{F}_{n-3}^{[i]} \mathcal{F}_{n-3}^{[i]} \mathcal{F}_{n-6}^{[i]} \mathcal{F}_{n-3}'^{[i]} \mathcal{F}_{n-3}'^{[i]}$.*
- d. *The $\mathcal{F}_n^{[i]}$ is symmetric.*
- e. *The scale factor between $\mathcal{F}_n^{[i]}$ and $\mathcal{F}_{n-3}^{[i]}$ is $1 + \sqrt{2}$.*

Proof. a. It is clear from the Proposition 5a, because 110 and 111 are not subwords of $\mathbf{f}^{[i]}$.

- b. By Proposition 3, we have $f_{n-1}^{[i+2]} = \varphi_i(f_n)$ for all integer $n \geq 2$ and $i \geq 0$. Moreover, φ_i maps the different segments as shown in the table 3.

For example in the figure 2, we show the mapping of f_{10} by φ_i when $i = 2, 3$.

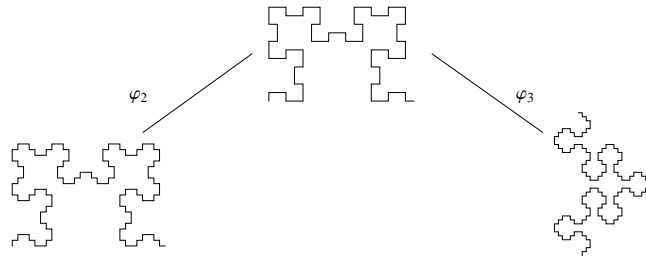


Figure 2: Mapping of $\varphi_2(f_{10})$ and $\varphi_3(f_{10})$.

If i is even		
$\varphi_i(01) = 0^{i+1}1$ 	$\varphi_i(10) = 0^i10$ 	$\varphi_i(00) = 00$
If i is odd		
$\varphi_i(01) = 0^{i+1}1$ 	$\varphi_i(10) = 0^i10$ 	$\varphi_i(00) = 00$

Table 3: Mapping of segments.

Hence, it is clear that φ_i preserves the geometric properties. By proposition 2 we have \mathcal{F}_n is similar to the curve \mathcal{F}_{n-3} then $\mathcal{F}_n^{[i]}$ is similar to $\mathcal{F}_{n-3}^{[i]}$.

c. It is clear from the Proposition 5e.

d. The proof runs like in b.

e. We show that

$$f_n^{[i]} = f_{n-3}^{[i]} f_{n-3}^{[i]} f_{n-6}^{[i]} l_{n-3}^{[i]} l_{n-3}^{[i]} = \Phi(f_{n-3}^{[i]}) ab \Phi(f_{n-3}^{[i]}) ab f_{n-6}^{[i]} \Phi(l_{n-3}^{[i]}) ba \Phi(l_{n-3}^{[i]}) ba.$$

Since ab is either 01 or 10, and $\mathcal{F}_n^{[i]} = \mathcal{F}_{n-3}^{[i]} \mathcal{F}_{n-3}^{[i]} \mathcal{F}_{n-6}^{[i]} \mathcal{F}_{n-3}^{[i]} \mathcal{F}_{n-3}^{[i]}$, then the first two curves are orthogonal and the last two curves are orthogonal. Let $L_n^{[i]}$ be the length of the curve $\mathcal{F}_n^{[i]}$ from first to last point drawn. Then $L_n^{[i]} = 2L_{n-3}^{[i]} + L_{n-6}^{[i]}$ and by definition, the scale factor Γ is

$$\Gamma = \frac{L_n^{[i]}}{L_{n-3}^{[i]}} = \frac{L_{n-3}^{[i]}}{L_{n-6}^{[i]}}$$

hence $\Gamma L_{n-3}^{[i]} = L_n^{[i]} = 2L_{n-3}^{[i]} + L_{n-6}^{[i]} = 2L_{n-3}^{[i]} + \frac{L_{n-3}^{[i]}}{\Gamma}$, then $\Gamma = 1 + \sqrt{2}$. □

Moreover, for each i the system $\mathcal{F}_n^{[i]}$ ($n \geq 0$) has as “attractor” the curve \mathcal{F} (the same argument given in the Proposition 6b).

Proposition 7. *The curve $\mathcal{F}_n^{[i]}$ display the Fibonacci word fractal pattern.*

4 Generalized Fibonacci Snowflakes

Recently the combinatorics on words has been used in modeling of problems of tessellations in the plane with polyominoes, (see e.g. [2], [3], [4], [7], [8], [9]). A path in the square lattice is a polygonal path made of the elementary unit translations

$$0 = (1, 0), \quad 1 = (0, 1), \quad 2 = (-1, 0), \quad 3 = (0, -1).$$

These paths are conveniently encoded by words on the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$. We say that a path w is closed if it satisfies $|w|_0 = |w|_2$ and $|w|_1 = |w|_3$. A simple path is a word w such that none of its proper subwords is a closed path. A boundary word is a closed path such that none of its proper subwords is closed. A polyomino is a subset of $\mathbb{Z} \times \mathbb{Z}$ contained in some boundary word.

Example 4. In the figure 3 we show a polyomino P , such that starting from point S (counterclockwise) the boundary $\mathbf{b}(P)$ is coded by the word $w = 2122323030103011$. Moreover, we denoted by \hat{w} the path traveled in the opposite direction, i.e., $\hat{w} = \rho^2(w^R)$, where ρ^2 is the morphism defined by $\rho^2(a) = 2 + a$, $a \in \mathcal{A}$. In this example $\hat{w} = \rho^2(1103010303232212) = 3321232121010030$.

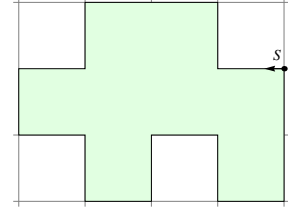


Figure 3: Polyomino P .

The problem of deciding if a given polyomino tiles the plane by translation was first considered by Wisjhoff and Van Leeuwen who coined the term exact polyomino for these [20]. Beaquier and Nivat [2] proved that a polyomino P tiles the plane by translations if and only if the boundary word $\mathbf{b}(P)$ is equal up to a cyclic permutation of the symbols to $A \cdot B \cdot C \cdot \hat{A} \cdot \hat{B} \cdot \hat{C}$, where one of the variables in the factorization may be empty. This condition is referred as the BN-factorization [8]. If the boundary word is equal to $A \cdot B \cdot C \cdot \hat{A} \cdot \hat{B} \cdot \hat{C}$ such a polyomino is called pseudo-hexagon.

Example 5. The polyomino in the figure 4 (left) is an exact polyomino and its boundary can be factorized by $122 \cdot 212 \cdot 323 \cdot 003 \cdot 030 \cdot 101$, (the factorization is not necessarily in a unique way).

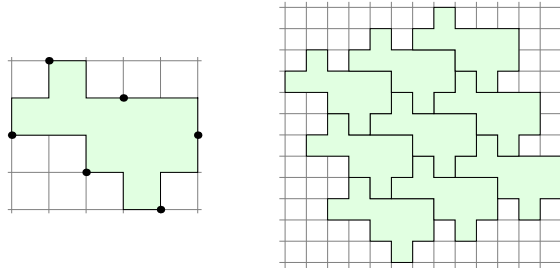


Figure 4: Exact polyomino and tiling.

When one of the variables is empty, i.e., $\mathbf{b}(P) = A \cdot B \cdot \hat{A} \cdot \hat{B}$, we say that P is a square polyomino. In [4], they prove that an exact polyomino tiles the plane in at most two distinct ways. Squares polyominoes having exactly two distinct BN-factorizations are called double squares. For instance, Christoffel and Fibonacci tiles, introduced recently [3], are examples of double squares, however, there exist double squares not in the Christoffel and Fibonacci tile families. In [7], they study the combinatorial properties and the problem of generating exhaustively double square tiles, however, they did not study the geometric properties, only in the case of Fibonacci polyominoes [6].

In this section, we study a new generalization of Fibonacci polyominoes from i -Fibonacci words. We use the same procedure as in [3] and we present some geometric properties.

4.1 Construction of Generalized Fibonacci Polyominoes

First, rewrite the i -Fibonacci words over alphabet $\{0, 2\} \subset \mathcal{A}$, specifically we apply the morphism $0 \rightarrow 2, 1 \rightarrow 0$. Next, apply the operator Σ_1 followed by the operator Σ_0 , where

$$\Sigma_\alpha(w) = \alpha \cdot (\alpha + w_1) \cdot (\alpha + w_1 + w_2) \cdots (\alpha + w_1 + w_2 + \cdots + w_n),$$

with $\alpha \in \mathcal{A}$ and $w = w_1 w_2 \cdots w_n$. This yield the words $\mathbf{p}^{[i]} = \Sigma_0 \Sigma_1 \mathbf{f}^{[i]}$.

Example 6. In the table 4, we show the first words $\mathbf{p}^{[i]}$, with its corresponding curves. If $n = 2$ corresponds to a version of the Fibonacci word fractal, with only segments of length 1, [3].

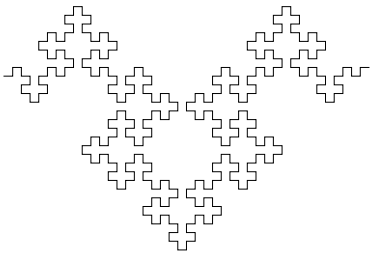
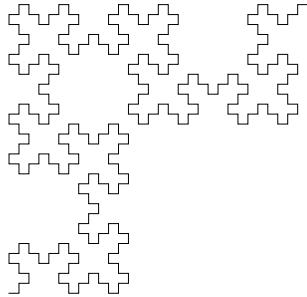
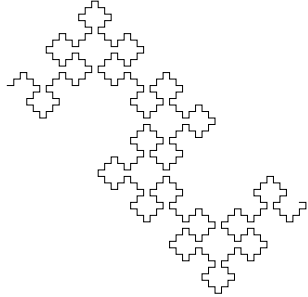
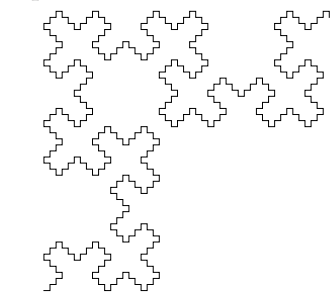
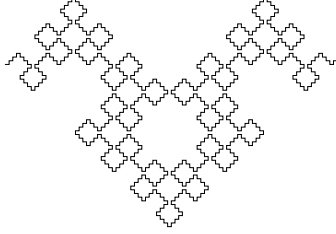
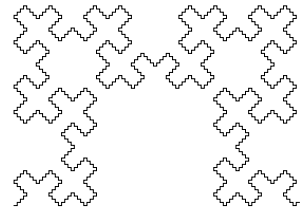
$\mathbf{p}^{[1]} = \mathbf{p}^{[2]} = 010303230301 \dots$ 	$\mathbf{p}^{[3]} = 01012121010303 \dots$ 	$\mathbf{p}^{[4]} = 01010303032323 \dots$ 
$\mathbf{p}^{[5]} = 01010121212101 \dots$ 	$\mathbf{p}^{[6]} = 01010103030303 \dots$ 	$\mathbf{p}^{[7]} = 01010101212121 \dots$ 

Table 4: Words $\mathbf{p}^{[i]}$ and its corresponding curves.

Given a word $w \in \mathcal{A}^*$, we define the word $\Delta(w) = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \in \mathcal{A}^*$, then it is clear that $\Delta(\mathbf{p}^{[i]}) = \Sigma_1 \mathbf{f}^{[i]}$. We shall denote this sequence by $\mathbf{q}^{[i]}$. Last, we define the morphism \overline{a} , with $a \in \mathcal{A}$, as $\overline{0} = 0, \overline{1} = 3, \overline{2} = 2, \overline{3} = 1$. Moreover, the words $w \in \mathcal{A}^*$ satisfying $\overline{w} = w^R$ are called antipalindromes.

Definition 11. Consider the sequence $\left\{ q_n^{[i]} \right\}_{n \geq 0}$ defined by:

- If i is even, $q_0^{[i]} = \epsilon, q_1^{[i]} = 1, q_2^{[i]} = (13)^{\frac{i}{2}}$ and

$$q_n^{[i]} = \begin{cases} q_{n-1}^{[i]} q_{n-2}^{[i]}, & n \equiv 1 \pmod{3} \\ q_{n-1}^{[i]} q_{n-2}^{[i]}, & n \equiv 0, 2 \pmod{3} \end{cases}$$

- If i is odd, $q_0^{[i]} = \epsilon, q_1^{[i]} = 1, q_2^{[i]} = (13)^{\frac{i-1}{2}} 1$ and

$$q_n^{[i]} = \begin{cases} q_{n-1}^{[i]} q_{n-2}^{[i]}, & n \equiv 0 \pmod{3} \\ q_{n-1}^{[i]} q_{n-2}^{[i]}, & n \equiv 1, 2 \pmod{3} \end{cases}$$

It is clear that $|q_n^{[i]}| = F_{n-1}^{[i]}$.

Example 7. The first terms of $\{q_n^{[i]}\}_{n \geq 0}$ are:

$$\begin{aligned}\{q_n^{[2]}\}_{n \geq 0} &= \{\epsilon, 1, 13, 133, 13313, 13313311, 1331331131131, \dots\}, \\ \{q_n^{[3]}\}_{n \geq 0} &= \{\epsilon, 1, 131, 1311, 1311313, 13113133133, 13113133133131313, \dots\}, \\ \{q_n^{[4]}\}_{n \geq 0} &= \{\epsilon, 1, 1313, 13133, 131331313, 13133131331311, \dots\}, \\ \{q_n^{[5]}\}_{n \geq 0} &= \{\epsilon, 1, 13131, 131311, 13131131313, 1313113131331313, \dots\}.\end{aligned}$$

The following propositions generalize the case when $i = 2$, [3].

Proposition 8. The word $\mathbf{q}^{[i]}$ is the limit of the sequence $\{q_n^{[i]}\}_{n \geq 0}$.

Proof. We know that $\Delta(\mathbf{q}^{[i]}) = \mathbf{f}^{[i]}$, then it suffices to prove that $\Delta(q_n^{[i]})\alpha_{n-1} = f_{n-1}^{[i]}$ for all $n \geq 2$, where $\alpha_n = 2$ if n is even and $\alpha_n = 0$ if n is odd. By induction on n . If i is even, then

$$\begin{aligned}\Delta(q_2^{[i]})\alpha_1 &= \Delta((13)^{i/2})\alpha_1 = (22)^{i/2-1}20 = 2^{i-2}20 = 2^{i-1}0 = f_1^{[i]}, \\ \Delta(q_3^{[i]})\alpha_2 &= \Delta((13)^{i/2}3)\alpha_2 = 2^{i-1}02 = f_1^{[i]}f_0^{[i]} = f_2^{[i]}, \\ \Delta(q_4^{[i]})\alpha_3 &= \Delta((13)^{i/2}3(13)^{i/2})\alpha_3 = 2^{i-1}02i0 = f_2^{[i]}f_1^{[i]} = f_3^{[i]}.\end{aligned}$$

Assume for all m , with $2 \leq m < n$; we prove it for n . We only prove the case $n \cong 1 \pmod 3$, since the argument is similar for the other cases. Let $n = 3k + 1$ for some integer k . Then

$$\Delta(q_{3k+1}^{[i]})\alpha_{3k} = \Delta(q_{3k}^{[i]}q_{3k-1}^{[i]})\alpha_{3k} = \Delta(q_{3k}^{[i]})\alpha_{3k-1}\Delta(q_{3k-1}^{[i]})\alpha_{3k-2} = f_{3k-1}^{[i]}f_{3k-2}^{[i]} = f_{3k}^{[i]}.$$

If i is odd, the prove is similar. \square

Proposition 9. Let $n \in \mathbb{N}$ and $\sigma_n = 1$ if n is even and $\sigma_n = 3$ if n is odd. Then if i is even $q_{3n+1}^{[i]} = r\sigma_n$, $q_{3n+2}^{[i]} = m\overline{\sigma_n}$ and $q_{3n+3}^{[i]} = p\overline{\sigma_n}$ for some antipalindrome p , some palindromes r , m . If i is odd $q_{3n+1}^{[i]} = r\overline{\sigma_n}$, $q_{3n+2}^{[i]} = m\overline{\sigma_n}$ and $q_{3n+3}^{[i]} = p\sigma_n$ for some antipalindrome m , some palindromes r , p .

Proof. The proof is by induction on n . If i is even, for $n = 0$ we have $q_1^{[i]} = \epsilon \cdot 1$, $q_2^{[i]} = (13)^{i/2} = ((13)^{\frac{i}{2}-1}1)3 = ((13)^{\frac{i}{2}-1}1) \cdot \overline{1}$ and $q_3^{[i]} = (13)^{i/2}3 = (13)^{i/2} \cdot \overline{1}$. Now, suppose that $q_{3n+1} = r\sigma_n$, $q_{3n+2} = m\overline{\sigma_n}$ and $q_{3n+3} = p\overline{\sigma_n}$ for some antipalindrome p , some palindromes r , m . Then

$$\begin{aligned}q_{3n+4}^{[i]} &= q_{3n+3}^{[i]}q_{3n+2}^{[i]} = q_{3n+2}^{[i]}\overline{q_{3n+1}^{[i]}}q_{3n+2}^{[i]} = m\overline{\sigma_n} \cdot \overline{r\sigma_n} \cdot m\overline{\sigma_n} = m\overline{\sigma_n}r\overline{\sigma_n}m \cdot \sigma_{n+1}, \\ q_{3n+5}^{[i]} &= q_{3n+4}^{[i]}q_{3n+3}^{[i]} = q_{3n+3}^{[i]}q_{3n+2}^{[i]}\overline{q_{3n+1}^{[i]}} = p\overline{\sigma_n} \cdot m\overline{\sigma_n} \cdot \overline{p\overline{\sigma_n}} = p\overline{\sigma_n}m\overline{\sigma_n}p \cdot \overline{\sigma_{n+1}}, \\ q_{3n+6}^{[i]} &= q_{3n+5}^{[i]}q_{3n+4}^{[i]} = q_{3n+4}^{[i]}q_{3n+3}^{[i]}\overline{q_{3n+2}^{[i]}} = m\overline{\sigma_n}r\overline{\sigma_n}m\overline{\sigma_n} \cdot \overline{p\overline{\sigma_n}} \cdot \overline{m\overline{\sigma_n}r\overline{\sigma_n}m} \\ &= m\overline{\sigma_n}r\overline{\sigma_n}m\overline{\sigma_n}p\overline{\sigma_n}m\overline{\sigma_n}r\overline{\sigma_n}m \cdot \overline{\sigma_{n+1}}\end{aligned}$$

with palindromes $m\overline{\sigma_n}r\overline{\sigma_n}m$ and $p\overline{\sigma_n}m\overline{\sigma_n}p$, and antipalindrome $m\overline{\sigma_n}r\overline{\sigma_n}m\overline{\sigma_n}p\overline{\sigma_n}m\overline{\sigma_n}r\overline{\sigma_n}m$. If i is odd, the prove is similar. \square

Proposition 10. Let n be a positive integer and $\alpha \in \mathcal{A}$ then

- a) The path $\Sigma_\alpha q_n^{[i]}$ is simple.
- b) If i is even, then the path $\Sigma_\alpha^\circ(q_{3n}^{[i]})^4$ is the boundary word of a polyomino.
- c) If i is odd, then the path $\Sigma_\alpha^\circ(q_{3n+2}^{[i]})^4$ is the boundary word of a polyomino.

Where $\Sigma_\alpha^\circ(w) = \alpha \cdot (\alpha + w_1) \cdot (\alpha + w_1 + w_2) \cdots (\alpha + w_1 + w_2 + \cdots w_{n-1})$.

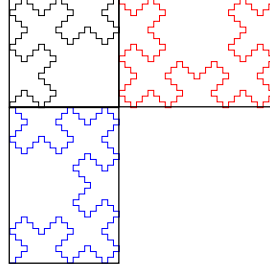


Figure 5: $\Sigma_0 q_{12}^{[5]}$ is divide into parts $\Sigma_0 q_{10}^{[5]}$, $\Sigma_2 q_9^{[5]}$ and $\Sigma_3 q_{10}^{[5]}$.

Proof. a) The proof is by induction on n . It is the similar to [3] or [6], we only describe the basic ideas because the proof is rather technical. For $n = 1, 2, 3$ it is clear. Assume for all j such that $1 \leq j < n$; we prove it for n . The idea is to divide the path $\Sigma_\alpha q_n^{[i]}$ into three smaller parts, for example the path $\Sigma_0 q_{12}^{[5]}$ is divide into parts $\Sigma_0 q_{10}^{[5]}$, $\Sigma_2 q_9^{[5]}$ and $\Sigma_3 q_{10}^{[5]}$, (see figure 5).

By the induction hypothesis $\Sigma_{\alpha_1} q_{n-2}^{[i]}$ and $\Sigma_{\alpha_2} q_{n-3}^{[i]}$ are simples, moreover, the three smaller paths are contained in disjoint boxes, then $\Sigma_{\alpha_1} q_{n-2}^{[i]}$ is simple.

b) If i is even. From the Proposition 9, we have $q_{3n}^{[i]} = p \overline{\sigma_{n-1}}$ for some antipalindrome $p = w_1 \cdots w_n$ and $\overline{\sigma_{n-1}} \in \{1, 3\}$. If $\overline{\sigma_{n-1}} = 3$, we can consider the reversal of the path, so suppose that $\overline{\sigma_{n-1}} = 1$. Hence $\Sigma_\alpha^\circ (q_{3n}^{[i]})^4 = \Sigma_\alpha(p1 \cdot p1 \cdot p1 \cdot p)$, as

$$\Sigma_\alpha p1 = \alpha \cdot (\alpha + w_1) \cdot (\alpha + w_1 + w_2) \cdots (\alpha + w_1 + w_2 + \cdots w_n + 1)$$

and $|p|_1 = |p|_3$, because p is an antipalindrome, then

$$\alpha + w_1 + w_2 + \cdots w_n + 1 = \alpha + |p|_1 + 3|p|_3 + 1 = \alpha + 4|p|_1 + 1 \cong \alpha + 1 \pmod{4}$$

Therefore

$$\Sigma_\alpha^\circ (q_{3n}^{[i]})^4 = \Sigma_\alpha(p1 \cdot p1 \cdot p1 \cdot p) = \Sigma_\alpha p \cdot \Sigma_{\alpha+1} p \cdot \Sigma_{\alpha+2} p \cdot \Sigma_{\alpha+3} p.$$

But, the initial segments in the paths $\Sigma_\alpha p$ and $\Sigma_{\alpha+1} p$ are orthogonal because α and $\alpha + 1$ represent orthogonal vectors. Hence $\Sigma_\alpha p \cdot \Sigma_{\alpha+1} p \cdot \Sigma_{\alpha+2} p \cdot \Sigma_{\alpha+3} p$ is a closed polygonal path, illustrated in the figure 6 with an angle of $+\pi/2$ counterclockwise.

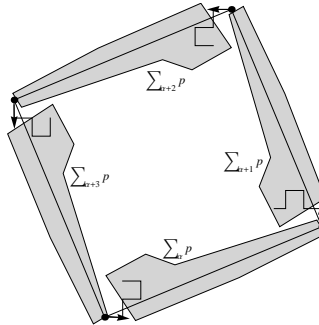


Figure 6: Case b) with an angle of $+\pi/2$.

c) If i is odd, the prove is similar.

□

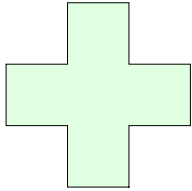
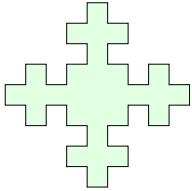
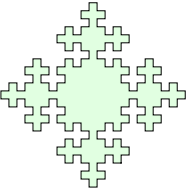
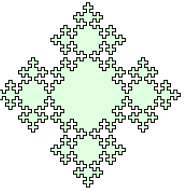
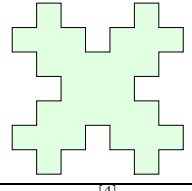
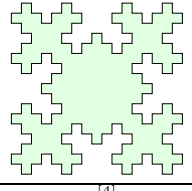
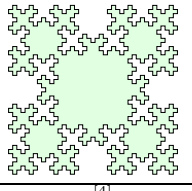
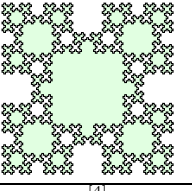
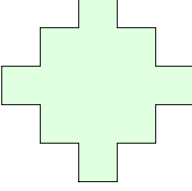
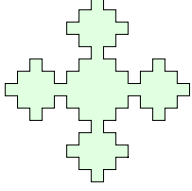
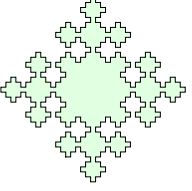
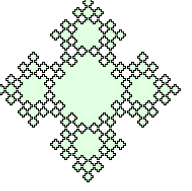
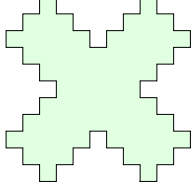
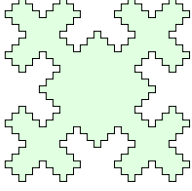
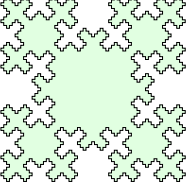
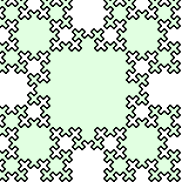
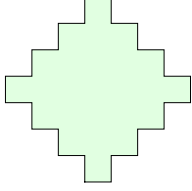
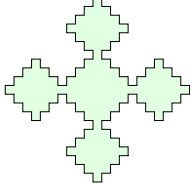
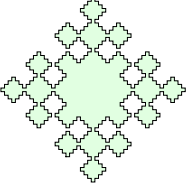
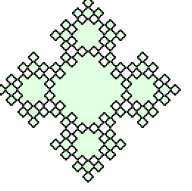
$\Pi_1^{[2]}$ 	$\Pi_2^{[2]}$ 	$\Pi_3^{[2]}$ 	$\Pi_4^{[2]}$ 
$\Pi_1^{[3]}$ 	$\Pi_2^{[3]}$ 	$\Pi_3^{[3]}$ 	$\Pi_4^{[3]}$ 
$\Pi_1^{[4]}$ 	$\Pi_2^{[4]}$ 	$\Pi_3^{[4]}$ 	$\Pi_4^{[4]}$ 
$\Pi_1^{[5]}$ 	$\Pi_2^{[5]}$ 	$\Pi_3^{[5]}$ 	$\Pi_4^{[5]}$ 
$\Pi_1^{[6]}$ 	$\Pi_2^{[6]}$ 	$\Pi_3^{[6]}$ 	$\Pi_4^{[6]}$ 

Table 5: The i -Generalized Fibonacci Snowflakes $\Pi_n^{[i]}$, for $i = 2, 3, 4, 5, 6$ and $n = 1, 2, 3, 4$.

An i -generalized Fibonacci snowflake of order n is a polyomino having $\Sigma_\alpha^\circ(q_{3n}^{[i]})^4$ or $\Sigma_\alpha^\circ(q_{3n+2}^{[i]})^4$ as a boundary word, we denote this as $\Pi_n^{[i]}$. In the table 5 we show first i -generalized Fibonacci snowflakes.

Theorem 2. *The i -generalized Fibonacci snowflake of order $n \geq 1$ is a double square, for all integer positive i .*

Proof. Suppose that i even. We show in the Proposition 10–b) that

$$\Sigma_\alpha^\circ(q_{3n}^{[i]})^4 = \Sigma_\alpha(p1 \cdot p1 \cdot p1 \cdot p) = \Sigma_\alpha p \cdot \Sigma_{\alpha+1} p \cdot \Sigma_{\alpha+2} p \cdot \Sigma_{\alpha+3} p.$$

Moreover $w_j = -w_{n-(j-1)}$, for all j with $1 \leq j \leq n$, because p is an antipalindrome. Then

$$\begin{aligned} \Sigma_{\alpha+2} p &= (\alpha + 2)(\alpha + 2 + w_1) \cdots (\alpha + 2 + w_1 + w_2 + \cdots w_n) \\ &= (\alpha + 2 + w_1 + w_2 + \cdots w_n)(\alpha + 2 + w_1 + w_2 + \cdots w_{n-1}) \cdots (\alpha + 2) \\ &= \widehat{\Sigma_\alpha p}. \end{aligned}$$

Hence

$$\Sigma_\alpha^\circ(q_{3n}^{[i]})^4 = \Sigma_\alpha p \cdot \Sigma_{\alpha+1} p \cdot \Sigma_{\alpha+2} p \cdot \Sigma_{\alpha+3} p = \Sigma_\alpha p \cdot \Sigma_{\alpha+1} p \cdot \widehat{\Sigma_\alpha p} \cdot \widehat{\Sigma_{\alpha+1} p}.$$

By the other hand, the word $q_{3n}'^{[i]} = \overline{q_{3n-2}^{[i]} q_{3n-1}^{[i]}}$ corresponds to another boundary word of the same title. In fact, by the Proposition 9, we have $q_{3n-1}^{[i]} = m1$ and $q_{3n-2}^{[i]} = r3$, for some palindromes m and r . Hence $p1 = q_{3n}^{[i]} = q_{3n-1}^{[i]} \overline{q_{3n-2}^{[i]}} = m1\bar{r}1$, so that $p = m1\bar{r}$. Therefore

$$q_{3n}'^{[i]} = \overline{q_{3n-2}^{[i]} q_{3n-1}^{[i]}} = \bar{r}1m1 = p^R 1 = \bar{p}1$$

$$\text{and } \Sigma_\alpha^\circ(q_{3n}'^{[i]})^4 = \Sigma_\alpha(\bar{p}1 \cdot \bar{p}1 \cdot p^R 1 \cdot p^R) = \Sigma_\alpha \bar{p} \cdot \Sigma_{\alpha+1} \bar{p} \cdot \widehat{\Sigma_\alpha \bar{p}} \cdot \widehat{\Sigma_{\alpha+1} \bar{p}}.$$

□

Remark. Note that if $A \cdot B \cdot \hat{A} \cdot \hat{B}$ is a BN-factorization of an i -generalized Fibonacci snowflake, then A and B are palindromes, because p is an antipalindrome then $\Sigma_\alpha p$ and $\Sigma_\alpha \bar{p}$ are palindromes.

Example 8. In the table 6, we show tessellations of $\Pi_2^{[3]}$ and $\Pi_3^{[6]}$.

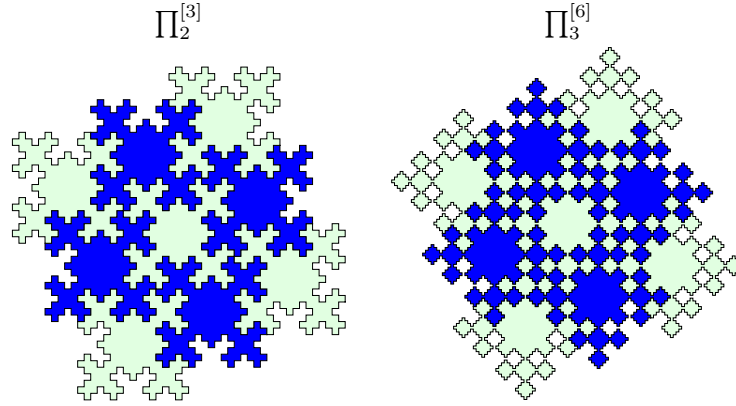


Table 6: Tessellations of $\Pi_2^{[3]}$ and $\Pi_3^{[6]}$.

4.2 Some Geometric Properties

Definition 12. The number $P^{[i]}(n)$ is defined recursively by $P^{[i]}(0) = -i$, $P^{[i]}(1) = i + 1$ and $P^{[i]}(n) = 2P^{[i]}(n-1) + P^{[i]}(n-2)$ for all $n \geq 2$ and $i \geq 0$.

For $i = 0$ we have the Pell numbers. In the table 7 are the first numbers $P^{[i]}(n)$.

i	$P^{[i]}(n)$
0	$\{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \dots\}$, (A000129).
1	$\{-1, 2, 3, 8, 19, 46, 111, 268, 647, 1562, 3771, \dots\}$, (A078343).
2	$\{-2, 3, 4, 11, 26, 63, 152, 367, 886, 2139, 5164, \dots\}$.
3	$\{-3, 4, 5, 14, 33, 80, 193, 466, 1125, 2716, 6557, \dots\}$.
4	$\{-4, 5, 6, 17, 40, 97, 234, 565, 1364, 3293, 7950, \dots\}$.
5	$\{-5, 6, 7, 20, 47, 114, 275, 664, 1603, 3870, 9343, \dots\}$.

Table 7: First numbers $P^{[i]}(n)$.

Proposition 11. A formula for the $P^{[i]}(n)$ numbers is

$$P^{[i]}(n) = \frac{1}{4} \left(\left(1 + \sqrt{2}\right)^n (\sqrt{2} - (2 - 2\sqrt{2})i) - \left(1 - \sqrt{2}\right)^n (\sqrt{2} + (2 + 2\sqrt{2})i) \right).$$

Proof. The result clearly holds by induction on n . \square

Let $\alpha \in \mathcal{A}$, we denote by $\vec{\Sigma}_\alpha q$ the coordinates of the vector whose initial point is the origin and the terminal point is the last point of the path $\Sigma_\alpha^\circ(q)$. In the next proposition, we show that the coordinates of the vector $\vec{\Sigma}_0(q_n^{[i]})$ are expressed in terms of the numbers $P^{[i]}(n)$. A similar thing happens when $\alpha = 1, 2, 3$.

Proposition 12. *For all $n \in \mathbb{N}$, we have that if i is even then*

$$\begin{aligned}\vec{\Sigma}_0 q_{3n+1}^{[i]} &= \left(P^{[k]}(n+1) + P^{[k]}(n), 0 \right), \\ \vec{\Sigma}_0 q_{3n+2}^{[i]} &= \left(P^{[k]}(n+1), (-1)^n P^{[k]}(n+1) \right), \\ \vec{\Sigma}_0 q_{3n+3}^{[i]} &= \left(P^{[k]}(n+2), (-1)^n P^{[k]}(n+1) \right),\end{aligned}$$

where $k = \frac{i-2}{2}$. If i is odd then

$$\begin{aligned}\vec{\Sigma}_0 q_{3n+1}^{[i]} &= \begin{cases} (P^{[k]}(n+1) + P^{[k]}(n), 0), & \text{if } n \text{ is even,} \\ (0, P^{[k]}(n+1) + P^{[k]}(n)), & \text{if } n \text{ is odd,} \end{cases} \\ \vec{\Sigma}_0 q_{3n+2}^{[i]} &= \begin{cases} (P^{[k]}(n+2), P^{[k]}(n+1)), & \text{if } n \text{ is even,} \\ (P^{[k]}(n+1), P^{[k]}(n+2)), & \text{if } n \text{ is odd,} \end{cases} \\ \vec{\Sigma}_0 q_{3n+3}^{[i]} &= \left(P^{[k]}(n+2), P^{[k]}(n+2) \right)\end{aligned}$$

where $k = \frac{i-3}{2}$.

Proof. By induction on n . If i is even. For $n = 0$ it is clear. Assume for all j such that $0 \leq j \leq 3n+5$; we prove it for $3n+6$. Then passing to vectors we have

$$\begin{aligned}\vec{\Sigma}_0 q_{3n+6}^{[i]} &= \vec{\Sigma}_0 q_{3n+5}^{[i]} + \vec{\Sigma}_0 \overline{q_{3n+4}^{[i]}} \\ &= \left(P^{[k]}(n+2), (-1)^{n+1} P^{[k]}(n+2) \right) + \overline{\left(P^{[k]}(n+2) + P^{[k]}(n+1), 0 \right)} \\ &= \left(P^{[k]}(n+2), (-1)^{n+1} P^{[k]}(n+2) \right) + \left(P^{[k]}(n+2) + P^{[k]}(n+1), 0 \right) \\ &= \left(2P^{[k]}(n+2) + P^{[k]}(n+1), (-1)^{n+1} P^{[k]}(n+2) \right) \\ &= \left(P^{[k]}(n+3), (-1)^{n+1} P^{[k]}(n+2) \right)\end{aligned}$$

where $\vec{\Sigma}_0 \overline{q_n^{[i]}} = (\overline{A}, \overline{B})$ is the coordinate the last point of the path $\Sigma_\alpha^\circ(\overline{q}_n)$. In this case $\vec{\Sigma}_0 \overline{q_{3n+4}^{[i]}} = \vec{\Sigma}_0 q_{3n+4}^{[i]}$, because \overline{a} leaves the horizontal direction unchanged. The other cases are similar. \square

Example 9. In the table 8 are the endpoint coordinates of the paths $\Sigma_0^\circ(q_n^{[4]})$ and the figure 7 shows the coordinates.

n	0	1	2	3	4
$\Sigma_0^\circ(q_{3n+1}^{[4]})$	(1, 0)	(5, 0)	(11, 0)	(27, 0)	(65, 0)
$\Sigma_0^\circ(q_{3n+2}^{[4]})$	(2, 2)	(3, -3)	(8, 8)	(19, -19)	(46, 6)
$\Sigma_0^\circ(q_{3n+3}^{[4]})$	(3, 2)	(8, -3)	(19, 8)	(46, -19)	(111, 46)

Table 8: Coordinates of the path $\Sigma_0^\circ(q_n^{[4]})$.

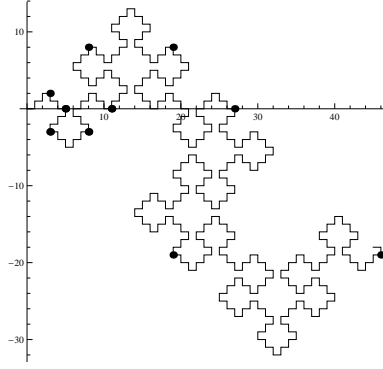


Figure 7: Graph with the coordinates of the path $\Sigma_0^\circ(q_n^{[4]})$.

The following proposition it is clear as $|q_n^{[i]}| = F_{n-1}^{[i]}$.

Proposition 13. *The perimeter $L(n, i)$, of the i -generalized Fibonacci snowflake of order n is*

$$L(n, i) = \begin{cases} 4F_{3n-1}^{[i]}, & \text{if } i \text{ is even,} \\ 4F_{3n+1}^{[i]}, & \text{if } i \text{ is odd.} \end{cases}$$

Proposition 14. *The area $A(n, i)$ of the i -generalized Fibonacci snowflake of order n is*

- a) *If i is even then $A(n, i) = (P^{[k]}(n+1))^2 + (P^{[k]}(n))^2$, where $k = \frac{i-2}{2}$.*
- b) *If i is odd then $A(n, i) = (P^{[k]}(n+2))^2 + (P^{[k]}(n+1))^2$, where $k = \frac{i-3}{2}$.*
- c) *$A(n, i)$ satisfies the recurrence formula $A(n, i) = 6A(n-1, i) - A(n-2, i)$ for all $n \geq 3$, (initial values can be calculated with the above items).*

Proof. Suppose that i is even. If a word $w \in \mathcal{A}^*$ is an antipalindrome then its corresponding polygonal line is symmetric with respect to midpoint of the vector $\vec{\Sigma}_\alpha w$, see lemma 2.6 in [6]. Moreover, from the Proposition 10-b), we have that the parallelogram determined by the word $\Sigma_\alpha^\circ(q_{3n}^{[i]})^4$ is a square, (in the figure 8, we show some examples for $i = 2, 3, 4$ and $n = 2$), and by the Proposition 12, the area $A(n, i)$ is equal to the area of square determined by $\Sigma_0^\circ(q_{3n}^{[i]}) = (P^{[k]}(n+1), \pm P^{[k]}(n))$. Hence $A(n, i) = (P^{[k]}(n+1))^2 + (P^{[k]}(n))^2$, where $k = \frac{i-2}{2}$. If i is odd, the proof is similar.

The case (c) is obtained from (a) and (b), and by the definition of $P^{[i]}(n)$.

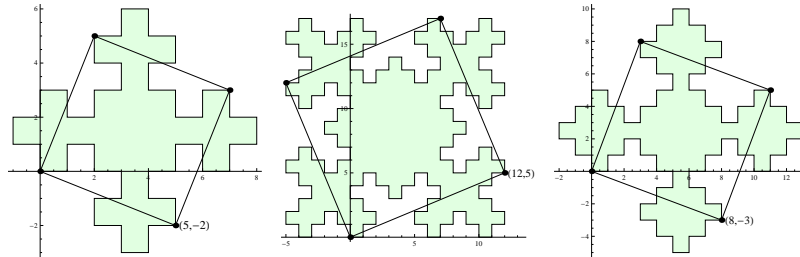


Figure 8: Examples, Areas of i -generalized Fibonacci snowflakes.

□

Let $S^{[i]}(n)$ be the smallest square having sides parallel to the axes and containing to $\Pi^{[i]}$. In the figure 9, we show the cases for $i = 4$ and $n = 2, 3$. If i is even, we have that $(A, B) = (P^{[i]}(n), (-1)^n P(n+1)^{[i]})$ from the proposition 12, therefore

$$S^{[i]}(n) = \left(\frac{A+3B}{2} - \frac{A-B}{2} - 1 \right)^2 = (2B-1)^2 = (2P^{[i]}(n+1)-1)^2$$

When i is odd is the same.

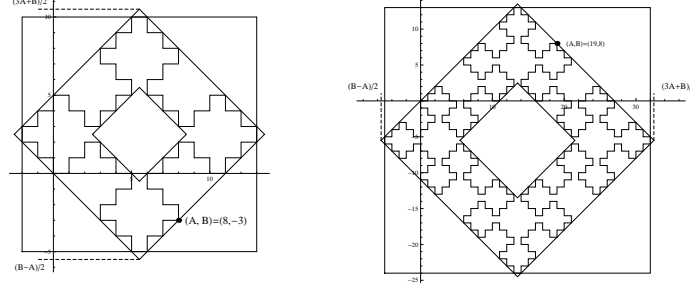


Figure 9: $S^{[i]}(n)$ for $i = 4$ and $n = 2, 3$.

The next theorem generalizes the theorem 1 of [5].

Theorem 3. *The fractal dimension of $\Pi^{[i]} = \lim_{n \rightarrow \infty} \Pi_n^{[i]}$ is*

$$\frac{3 \ln \phi}{\ln(1 + \sqrt{2})}.$$

Proof. Suppose that i is even, then the polyomino $\Pi_n^{[i]}$ is composed of $4|q_{3n}^{[i]}|$ unit segments and this value blows up when $n \rightarrow \infty$. However, the normalized polyomino $\frac{1}{2P^{[i]}(n+1)-1} \Pi_n^{[i]}$ stays bounded. It has $4|q_{3n}^{[i]}|$ segments of length $\frac{1}{2P^{[i]}(n+1)-1}$. Hence the total d -dimensional normalized polyomino has length

$$\frac{4|q_{3n}^{[i]}|}{(2P^{[i]}(n+1)-1)^d}$$

and therefore the fractal dimension of $\Pi^{[i]}$ is

$$d = \lim_{n \rightarrow \infty} \frac{\ln(4|q_{3n}^{[i]}|)}{\ln(2P^{[i]}(n+1)-1)} = \frac{3 \ln \phi}{\ln(1 + \sqrt{2})}.$$

□

5 Conclusion

In this article, we study a generalization of the Fibonacci word and the Fibonacci word fractal founds in [16]. Particularly, we defined the curves $\mathcal{F}^{[i]}$ from the i -Fibonacci words and show their properties remain. Moreover, the i -generalized Fibonacci snowflakes generalize the Fibonacci snowflake studied in [6] and they are a subclass of double squares. Finally, we found that the i -generalized Fibonacci snowflakes are related with the Fibonacci and Pell numbers, and some generalizations.

References

- [1] Allouche, J.-P., Shallit, J., *Automatic Sequences*, Cambridge University Press, Cambridge, 2003.
- [2] Beauquier, D., Nivat, M., “On translating one polyomino to tile the plane”, *Discrete & Computational Geometry* 6 (1991) 575-592.
- [3] Blondin-Massé, A., Brlek, S., Garon, A., Labbé, S., “Two infinite families of polyominoes that tile the plane by translation in two distinct ways”, *Theoretical Computer Science* 412 Issue 36 (2011) 4778-4786.
- [4] Blondin-Massé, A., Brlek, S., Labbé, S., “A parallelogram tile fills the plane by translation in at most two distinct ways”, *Discrete Applied Mathematics* 160 Issues 7-8 (2012) 1011-101.
- [5] Blondin-Massé, A., Brlek, S., Labbé, S., Mendès France, M., “Complexity of the Fibonacci snowflake”, eprint arXiv:1103.6171, (2011).
- [6] Blondin-Massé, A., Brlek, S., Labbé, S., Mendès France, M., “Fibonacci snowflakes”, Special Issue dedicated to Paulo Ribenboim, *Annales des Sciences Mathématiques du Québec* 35, No 2 (2011).
- [7] Blondin Massé, A., Garon, A., Labbé, S., “Combinatorial properties of double square tiles”, submitted to *Theoretical Computer Science*, accepted, 2012.
- [8] Brlek, S., “Interactions between Digital Geometry and Combinatorics on Words”, *Proceedings 8th Int. Conf. on Words 2011 (WORDS 2011)*, Prague, Czech Republic, September 12-16, *Electronic Proceedings in Theoretical Computer Science TCS* 63 (2011) 1-12.
- [9] Brlek, S., Fédou, J.-M, Provençal, X., “On the Tiling by Translation Problem”, *Discrete Applied Mathematics* 157 Issue 3 (2009) 464-475.
- [10] Cassaigne, J., “On extremal properties of the fibonacci word”, *RAIRO- Theor. Inf. Appl.* 42(4)(2008), 701-715.
- [11] Chuan, W., “Fibonacci words”, *Fibonacci Quart.*, 30.1 (1992), pp. 68-76.
- [12] Chuan, W., “Generating Fibonacci words”, *Fibonacci Quart.*, 33.2 (1995), pp. 104-112.
- [13] Chuan, W., “Extraction property of the golden sequence”, *Fibonacci Quart.*, 33.2 (1995), pp. 113-122.
- [14] Lothaire, M., *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2002.
- [15] Mignosi, F., Pirillo, G., “Repetitions in the Fibonacci infinite word”, *RAIRO Inform. Theor. Appl.*, 26 (1992), pp. 199-204.
- [16] Monnerot, A., “The Fibonacci Word Fractal”, preprint, <http://hal.archives-ouvertes.fr/hal-00367972/fr/>, (2009).
- [17] Prusinkiewicz, P., Lindenmayer, A., *The algorithmic beauty of plants*, Springer-Verlag. Nueva York, 2004. <http://algorithmicbotany.org/papers/abop/abop.pdf>.
- [18] Ramírez, J., Rubiano, G., “Generating fractals curves from homomorphisms between languages [with Mathematica®]” (Spanish), *Revista Integración, Escuela de Matemáticas, Universidad Industrial de Santander*, Vol. 30, No. 2, (2012), pp 129-150.
- [19] Sloane, N.J.A., *The On-Line Encyclopedia of Integer Sequences*. <https://oeis.org/>.
- [20] Wijshoff, H.A.G., van Leeuwen, J., “Arbitrary versus periodic storage schemes and tessellations of the plane using one type of polyomino”, *Information and Control* 62 (1984) 1-25.